Optimal Timing of Inventory Decisions with Price Uncertainty

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Introduction

When to Buy Inventory? A case of apparel

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Optimal Timing of Inventory Decisions

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JoS. A. Bank Suits Look Better Than Its Shares

Mounting inventories and softening sales are red flags for JoS. A. Bank.

By BILL ALPERT
Updated March 31, 2018 11:59 p.m. ET

THE SUITS AT JOS. A. BANK CLOTHIERS never go out of style. That's good, since some of them have been on the rack for a while. Defying industry wisdom (and Wall Street short sellers), the clothing chain has piles of inventory. 425 days' worth, at last report.

After many years of opening a new store every week, JoS. A. Bank now has some 420 stores across 42 states. Rival retailer Men's Wearhouse has 560 stores. The two chains may have saturated the market for mid-priced men's dress clothes: Inventory levels have been trending up at both companies in the last few years, while comparable-store sales growth has been trending down. Shares in Hampstead, Md.-based JoS. A. Bank (ticker: JOSB) have trended down, too, from 46 bucks last June to a recent 23. Like shares of Men's Wearhouse (MW), they trade at a multiple of less than 10-times earnings. Even so, bargain hunters should buy the suits and not the stock. Valuations in this industry have gone as low as six-times earnings. At that level, JoS. A. Bank shares would trade for 16.
When to Buy Inventory? A case of dairy

Lower prices, excess supply strain dairy farmers

In this April 30, 2015, photo, farm workers operate machinery in the milking parlor at Eldon Tweed Farm in West Charlton, N.Y. This is shaping up as a challenging year for all U.S. dairy farmers who enjoyed record high milk prices and low feed prices last year. AP Photo/Mike Groll

By Mary Esch, Associated Press
Introduction

When to Buy Inventory? A case of construction

China's Cooling Property Market May Risk Economic Growth

by Bloomberg News

October 12, 2016 – 5:00 PM EDT  Updated on October 12, 2016 – 11:08 PM EDT

High-profile tightening reverses two years of easing cycle
A property downturn increases potential for a hard-landing

Actions by China’s policy makers to rein in property prices in the bubble-prone nation may prove so effective that the economy’s growth rate could be affected next year.

At least 21 cities have introduced purchase restrictions and toughened mortgage lending since late September, reversing two years of easing to support home buyers. Goldman Sachs Group Inc. says more tightening is likely to follow if prices keep soaring, while Citigroup Inc. estimates shrinking demand may lead sales volume to contract in
Firm is a price taker. Demand and selling price are stochastic, forecasts are more accurate with time. Unit purchasing costs increase with time. One shot to buy inventory (or not buy at all).

- Is there an optimal order timing, and if so, is it deterministic or stochastic?
- What is the value of timing flexibility in the inventory procurement?
- How are the order timing and the value of timing flexibility affected by the volatility of price and demand?
Value of postponement

Real options

Optimal timing under cost, demand, and/or leadtime uncertainty:

Importantly, all previous real options papers work with smooth payoff functions using the McDonald and Siegel (1986) setup. To the best of our knowledge, there is no prior work on a real options problem with a newsvendor payoff function.
Price and demand

\[ P_t = \text{Price process}, \quad \frac{dP_t}{P_t} = h(m - \ln P_t)dt + s_p dz_p \]

\[ \xi_t = \text{Market size process}, \quad \frac{d\xi_t}{\xi_t} = \alpha \xi_t dt + s_\xi dz_\xi \]

\[ D_t = \text{Demand}, \quad D_T = P_T^{-\eta} \xi_T \]

\[ c_t = \text{Time-varying cost of purchasing inventory at time } t \] (may include holding cost for carrying inventory up to time \( T \))

\[ L = \text{Supplier's Leadtime}, \quad T - t \]
**Proposition**

The time $T$ value of the expected profit if optimal inventory decision is taken at time $s \geq t$ is given by:

$$Y(t, s) = E[P_T D_T | P_t, \epsilon_t] \int_{x_s}^{\infty} \Phi\left(d_{1^{**}} - \sigma_z\right) \frac{1}{\sqrt{2\pi}\sigma_x(t, s)} \exp\left[-\frac{(x_s - \mu_x(t, s) - k\sigma_x^2(t, s))^2}{2\sigma_x^2(t, s)}\right] \, dx_s.$$ 

Here, $d_{1^{**}}$ is the critical fractile for inventory decision taken at time $s$, $k$ is a constant as defined below, $x_s$ is the smallest value of the logarithm of price at which it becomes profitable to sell:

$$d_{1^{**}} = \Phi^{-1}\left(\max\left[0, 1 - \frac{c_s}{e^{-r(T-s)}E[P_T|P_s, \epsilon_s]}\right]\right), \quad \sigma_z = \sqrt{\eta^2\sigma_x^2 + \sigma_y^2 - 2\eta\sigma_{xy}}$$

$$k = (1 - \eta)e^{-h(T-s)} + \rho \frac{\sigma_y}{\sigma_x},$$

$$x_s = \left[\log c_s + r(T - s) - \frac{\sigma_x^2}{2} - (m - \frac{s_P^2}{2h})(1 - e^{-h(T-s)})\right] e^{h(T-s)}.$$

Parameters $\mu_x(t, s), \mu_y(t, s), \sigma_x(t, s)$ and $\sigma_y(t, s)$ are conditional mean/st.dev of $x_t = \ln P_t$ and $y_t = \ln \epsilon_t$. 

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**Model**

Fixed order time: Optimal order quantity and expected profit
Flexible order timing: Optimal stopping problem

- Let process \( \{W_t\} \) be the joint process of \( \{P_t\} \) and \( \{\epsilon_t\} \). Note that \( \{W_t\} \) is a Feller (Markov) process. The starting point \( W_0 = (P_0, \epsilon_0) \).

- Let \( \{\mathcal{F}_t^W\} \) be the natural filtration of \( W_t \). The stopping time \( \tau \) is adapted to \( \{\mathcal{F}_t^W\} \).

- Define the gain function:
  \[
  G(W_t) = E[P_T \min(q^*(t), D_T)|P_t, \epsilon_t] - c_t q^*(t)e^{r(T-t)} = \pi^*(t, P_t, \epsilon_t)e^{r(T-t)}.
  \]

- Define the value function
  \[
  V(W_t) = \sup_{t \leq \tau \leq T} E[G(W_\tau)],
  \]
  where the supremum is taken over all stopping times \( \tau \).

- \( G \) - continuous, and \( V \) - lower semi-continuous and bounded.

- To solve the optimal stopping problem (1) means: (i) find a stopping time \( \tau^* \) at which the supremum is attained, and (ii) compute the value function as explicitly as possible.
Define the continuation set:

\[ C = \{(t, W) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ : V(W_t) > G(W_t)\}, \]

and the stopping set:

\[ D = \{(t, W) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ : V(W_t) = G(W_t)\}. \]

Note that sets \( C \) and \( D \) split the state space into two disjoint subsets defined by deterministic functions \( G \) and \( V \).

Let \( \tau_D \) define the first hitting time for set \( D \):

\[ \tau_D = \inf\{t \geq 0, W_t \in D\}. \]

**Corollary 2.9, PS, p. 46**

Consider the optimal stopping problem (1) upon assuming that the corresponding boundedness condition is satisfied. Suppose that \( V \) is lsc and \( G \) is usc. Then \( \tau_D \) is optimal in (1).
Optimal stopping time is independent of the market size process

Consider the evolution of two processes that start at \((P_0, \epsilon_0)\) and \((P_0, \epsilon_1)\) at time 0. Let \(\epsilon_{t,0}\) and \(\epsilon_{t,1}\) be the values of \(\epsilon\) at time \(t\) in these two processes (correspondingly, \(D_{t,0}\) and \(D_{t,1}\)). Let \(G_0(t)\) and \(G_1(t)\) be the respective gain functions.

Then at time \(t\):

\[
\log \epsilon_{t,i} - \log \epsilon_i = (\alpha_\epsilon - \frac{1}{2} s_\epsilon^2) t + \alpha_\epsilon z_\epsilon(t), \quad i = 0, 1, \quad \text{and}
\]

\[
E[P_T D_{T,i}|P_t, \epsilon_{t,i}] = e^{\sigma_y^2/2+(1-\eta)\mu_x+(1-\eta)^2\sigma_x^2/2+(1-\eta)\sigma_{xy}+(\alpha_\epsilon+s_\epsilon^2)(T-t)\times \epsilon_{t,i}}, \quad \text{resulting in}
\]

\[
G_i(t) = G_{1-i}(t)\epsilon_i/\epsilon_{1-i}, \quad i = \{0, 1\}
\]

Thus,

\begin{enumerate}
  \item Gain function \(G\) scales perfectly with the initial condition;
  \item There is a one-to-one map between the continuation sets (stopping sets) defined above;
  \item There exists a one-to-one map between stopping times adapted to each of the two processes.
\end{enumerate}

Remaining question: characterize \(C\).
Define the characteristic operator of the two-dimensional non-time-homogeneous diffusion process \( \{x_t, y_t\} \), where \( x_t = \ln P_t \) and \( y_t = \ln \epsilon_t \).

\[
\hat{A}f(t, x, y) = \frac{\partial f}{\partial t} + h(m - \frac{s_p^2}{2h} - x_t) \frac{\partial f}{\partial x} + (\alpha \epsilon - \frac{s_\epsilon^2}{2}) \frac{\partial f}{\partial y} + \frac{s_p^2}{2} \frac{\partial^2 f}{\partial x^2} + \frac{s_\epsilon^2}{2} \frac{\partial^2 f}{\partial y^2} + s_p s_\epsilon \rho \frac{\partial^2 f}{\partial x \partial y}.
\]

The set \( U(t) = \{\hat{A}G(t, x, y) > 0\} \) is a subset of the continuation set at time \( t \), \( C(t) \) (Oksendal, 2003, p. 215). That is, continuing to wait is optimal for all \( (t, x, y) \) such that \( \hat{A}G(t, x, y) > 0 \), and buying inventory may be optimal if \( \hat{A}G(t, x, y) \leq 0 \).

**Relative incremental gain**

\[
\frac{\hat{A}G(t, x, y)}{G(t, x, y)} = \frac{\phi(d_1^* - \sigma_z)}{\phi(d_1^* - \sigma_z)} \left[ 1 - \Phi(d_1^*) \right] \left( -\frac{c_t'}{c_t} + r - \eta s_p^2 e^{-2h(T-t)} + \rho s_p s_\epsilon e^{-h(T-t)} + \frac{s_p^2}{2} \frac{1 - \Phi(d_1^*)}{\phi(d_1^*)} e^{-2h(T-t)} \sigma_z \right) - \frac{\partial \sigma_z}{\partial t},
\]

where \( \frac{\partial \sigma_z}{\partial t} = \left( 2\eta \rho s_p s_\epsilon e^{-h(T-t)} - \eta^2 s_p^2 e^{-2h(T-t)} - s_\epsilon^2 \right) / (2\sigma_z) \).

Note the relative incremental gain is independent of demand.
**Proposition**

The triplet \((t, P_t, \epsilon_t) \in C\) if

\[
\frac{c_t'}{c_t} - r + \eta s_p^2 e^{-2h(T-t)} - \rho s_p s_e e^{-h(T-t)} > 0 \text{ and one of the following conditions holds:}
\]

(i) Volatility condition:

\[
\frac{1 - \Phi(d_1^*)}{\phi(d_1^*)} s_p e^{-2h(T-t)} \sigma_z < \frac{c_t'}{c_t} - r + \eta s_p^2 e^{-2h(T-t)} - \rho s_p s_e e^{-h(T-t)}
\]

\[
- \left( \left( \frac{c_t'}{c_t} - r + \eta s_p^2 e^{-2h(T-t)} - \rho s_p s_e e^{-h(T-t)} \right)^2 + 2s_p^2 e^{-2h(T-t)} \sigma_z \frac{\partial \sigma_z}{\partial t} \right)^{1/2}
\]

(ii) Optionality condition:

\[
\frac{1 - \Phi(d_1^*)}{\phi(d_1^*)} s_p e^{-2h(T-t)} \sigma_z > \frac{c_t'}{c_t} - r + \eta s_p^2 e^{-2h(T-t)} - \rho s_p s_e e^{-h(T-t)}
\]

\[
+ \left( \left( \frac{c_t'}{c_t} - r + \eta s_p^2 e^{-2h(T-t)} - \rho s_p s_e e^{-h(T-t)} \right)^2 + 2s_p^2 e^{-2h(T-t)} \sigma_z \frac{\partial \sigma_z}{\partial t} \right)^{1/2}
\]
\( \hat{A}G(t, x, y)/G(t, x, y) \)

**Figure:** Plots of \( \hat{A}G/G \) as a function of \( d_1^* \) (left) and \( x \) (right). Parameter values: \( r - c'_t/c_t = -2.5, \ s_P = s_\epsilon = \eta = 1, \ h = 3, \ T - t = 0.04, \ \rho = 0.1. \)
Figure: Optimal upper (△) and lower (▽) log-price thresholds $x_t^*$. Cost trajectories are shown by the dashed lines. Dots show the span of the binary tree for the process $\{x_t\}$. Initial costs: (left) $c_0 = 1$, (center) $c_0 = 0.8$, (right) $c_0 = 1.2$.

Parameters:

$T = 0.25$, $m = \log P_0 = \alpha \epsilon = r = 0$, $s_P = s_\epsilon = 1$, $c'/c = 2.5$, $\rho = 0.1$, $\eta = 1$, $h = 3$, $\lambda = 0$. 
Figure: CDF of inventory order timing for $s_P = \{0.5, 1, \ldots, 2.5\}$ and the initial costs $c_0 = \{1, 0.8, 1.2\}$. Black dots show the optimal static ordering times for each $s_P$.

Parameters:

$T = 0.25, \ m = \log P_0 = \alpha = r = 0, \ s = 1, \ c'/c = 2.5, \ \rho = 0.1, \ \eta = 1, \ h = 3, \ \lambda = 0.$
Implications for ordering policy

Value of timing flexibility

**Table:** Performance of the time-flexible $V(W_0)$ vs. the best static-postponement $Y^*(0)$ ordering policies and the value of timing flexibility $VF^R$. The base case parameters:

$T = 0.25$, $m = \log P_0 = \alpha\epsilon = r = 0$, $s_P = s_\epsilon = 1$, $c'/c = 2.5$, $\rho = 0.1$, $\eta = 1$, $h = 3$, $\lambda = 0$.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$V(W_0)$</th>
<th>$c_0 = 1$</th>
<th>$VF^R$</th>
<th>$V(W_0)$</th>
<th>$c_0 = 0.8$</th>
<th>$VF^R$</th>
<th>$V(W_0)$</th>
<th>$c_0 = 1.2$</th>
<th>$VF^R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $s_P = 0.5$</td>
<td>0.001</td>
<td>0.001</td>
<td>50.7%</td>
<td>8.489</td>
<td>8.489</td>
<td>0.0%</td>
<td>0.000</td>
<td>0.000</td>
<td>0.0%</td>
</tr>
<tr>
<td>2 $s_P = 1$ (base case)</td>
<td>0.389</td>
<td>0.226</td>
<td>72.3%</td>
<td>6.713</td>
<td>6.713</td>
<td>0.0%</td>
<td>0.054</td>
<td>0.045</td>
<td>20.4%</td>
</tr>
<tr>
<td>3 $s_P = 1.5$</td>
<td>1.193</td>
<td>0.994</td>
<td>20.0%</td>
<td>4.407</td>
<td>4.407</td>
<td>0.0%</td>
<td>0.462</td>
<td>0.416</td>
<td>11.1%</td>
</tr>
<tr>
<td>4 $s_P = 2$</td>
<td>1.924</td>
<td>1.774</td>
<td>8.4%</td>
<td>3.935</td>
<td>3.394</td>
<td>15.9%</td>
<td>1.033</td>
<td>0.975</td>
<td>5.9%</td>
</tr>
<tr>
<td>5 $s_P = 2.5$</td>
<td>2.374</td>
<td>2.285</td>
<td>3.9%</td>
<td>4.029</td>
<td>3.815</td>
<td>5.6%</td>
<td>1.481</td>
<td>1.439</td>
<td>3.0%</td>
</tr>
<tr>
<td>6 $s_\epsilon = 0.5$</td>
<td>0.494</td>
<td>0.240</td>
<td>105.7%</td>
<td>9.204</td>
<td>9.204</td>
<td>0.0%</td>
<td>0.060</td>
<td>0.047</td>
<td>29.0%</td>
</tr>
<tr>
<td>7 $s_\epsilon = 2$</td>
<td>0.221</td>
<td>0.197</td>
<td>12.3%</td>
<td>2.620</td>
<td>2.620</td>
<td>0.0%</td>
<td>0.040</td>
<td>0.039</td>
<td>4.0%</td>
</tr>
<tr>
<td>8 $c'/c = 1$</td>
<td>2.413</td>
<td>2.398</td>
<td>0.7%</td>
<td>6.966</td>
<td>6.713</td>
<td>3.8%</td>
<td>0.822</td>
<td>0.817</td>
<td>0.6%</td>
</tr>
<tr>
<td>9 $c'/c = 4$</td>
<td>0.029</td>
<td>0.021</td>
<td>42.9%</td>
<td>6.713</td>
<td>6.713</td>
<td>0.0%</td>
<td>0.001</td>
<td>0.001</td>
<td>33.6%</td>
</tr>
<tr>
<td>10 $\rho = -0.5$</td>
<td>0.302</td>
<td>0.212</td>
<td>42.5%</td>
<td>4.887</td>
<td>4.887</td>
<td>0.0%</td>
<td>0.047</td>
<td>0.042</td>
<td>12.2%</td>
</tr>
<tr>
<td>11 $\rho = 0.5$</td>
<td>0.486</td>
<td>0.238</td>
<td>103.7%</td>
<td>8.661</td>
<td>8.661</td>
<td>0.0%</td>
<td>0.061</td>
<td>0.047</td>
<td>29.2%</td>
</tr>
<tr>
<td>12 $\eta = 0$</td>
<td>0.710</td>
<td>0.544</td>
<td>30.5%</td>
<td>7.872</td>
<td>7.872</td>
<td>0.0%</td>
<td>0.138</td>
<td>0.125</td>
<td>10.1%</td>
</tr>
<tr>
<td>13 $\eta = 2$</td>
<td>0.177</td>
<td>0.089</td>
<td>100.3%</td>
<td>4.626</td>
<td>4.626</td>
<td>0.0%</td>
<td>0.019</td>
<td>0.015</td>
<td>24.3%</td>
</tr>
<tr>
<td>14 $h = 0$</td>
<td>1.869</td>
<td>1.179</td>
<td>58.5%</td>
<td>6.521</td>
<td>6.521</td>
<td>0.0%</td>
<td>0.634</td>
<td>0.415</td>
<td>52.8%</td>
</tr>
<tr>
<td>15 $h = 6$</td>
<td>0.037</td>
<td>0.033</td>
<td>11.8%</td>
<td>7.112</td>
<td>7.112</td>
<td>0.0%</td>
<td>0.002</td>
<td>0.002</td>
<td>3.4%</td>
</tr>
<tr>
<td>16 $\lambda \rho_{PM} = \lambda \rho_{eM} = 0.5$</td>
<td>1.715</td>
<td>1.085</td>
<td>58.1%</td>
<td>5.755</td>
<td>5.755</td>
<td>0.0%</td>
<td>0.597</td>
<td>0.408</td>
<td>46.2%</td>
</tr>
<tr>
<td>17 $\lambda \rho_{PM} = \lambda \rho_{eM} = 1$</td>
<td>1.576</td>
<td>1.039</td>
<td>51.7%</td>
<td>5.079</td>
<td>5.079</td>
<td>0.0%</td>
<td>0.563</td>
<td>0.403</td>
<td>39.8%</td>
</tr>
<tr>
<td>18 Average</td>
<td>0.936</td>
<td>0.738</td>
<td>45.5%</td>
<td>6.093</td>
<td>6.034</td>
<td>1.5%</td>
<td>0.354</td>
<td>0.308</td>
<td>19.2%</td>
</tr>
</tbody>
</table>
Conclusions

- We address the question of the optimal time to buy inventory when demand and selling price are uncertain
  - Optimal stopping time depends on price only, and is independent of demand
  - It is optimal to wait when price is low
    - Optionality is valuable
  - Or when price is very high
    - Information is valuable

- Value of timing flexibility
  - Non-monotone in price volatility (unimodal)
  - Decreasing in demand volatility
  - Timing flexibility is valuable despite exponentially increasing purchasing costs
Two sentences summary of the paper:

(i) In the presence of price volatility, the time-flexible inventory policy leads to higher profit margins and creates an opportunity for the supplier and buyer firms in situations where profit margin is too low for them to transact under static postponement policies.

(ii) Purchase timing policy: wait when price is too low or too high, otherwise buy the optimal quantity given the current information.
Thank you!

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The paper is available upon request